On the relation between polynomial deformations of $sl(2,R)$ and quasi-exactly solvability

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Abstract

A general method based on the polynomial deformations of the Lie algebra $sl(2,R)$ is proposed in order to exhibit the quasi-exactly solvability of specific Hamiltonians implied by quantum physical models. This method using the finite-dimensional representations and differential realizations of such deformations is illustrated on the sextic oscillator as well as on the second harmonic generation.

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1 Introduction

In the late eighties, Turbiner and Ushveridze [1] discovered some cases where a finite number of eigenvalues (and the associated eigenfunctions) of the spectral problem for the Schrödinger operator

$$\bar{H}\psi = E\psi \ , \ \bar{H} = -\frac{d^2}{dy^2} + V(y) \ , \ y \in R \ or \ R^+ \tag{1}$$

can be found explicitely. The corresponding problems have been called quasi-exactly solvable (QES). Since that first step, QES equations have been classified [2] according to their relation with the finite-dimensional representations of the Lie algebra $sl2, R)$. Indeed QES Schrödinger equations as given in (1) can be written as

$$H\phi = E\phi \ , \ H = p_4(x)\frac{d^2}{dx^2} + p_3(x)\frac{d}{dx} + p_2(x) \ , \ x \in R \ or \ R^+ \tag{2}$$

through ad-hoc changes of variables and functions [2]

$$x = x(y) \ , \ \psi = exp(\chi)\phi, \tag{3}$$

if $p_j(x)(j = 2, 3, 4)$ refer to polynomials of order j in x. The Hamiltonian H in (2) can also be expressed in terms of the first-order differential operators

$$\begin{aligned} j_+ &= -x^2\frac{d}{dx} + 2jx, \\ j_0 &= x\frac{d}{dx} - j, \\ j_- &= \frac{d}{dx}, \quad j = 0, \frac{1}{2}, 1, ..., \end{aligned} \tag{4}$$

satisfying the $sl(2, R)$ commutation relations i.e.

$$[j_0, j_\pm] = \pm j_\pm, \tag{5}$$

$$[j_+, j_-] = 2j_0, \tag{6}$$

the Casimir operator of this structure being $C = j_+j_- + j_0^2 - j_0$. The operator H is then

$$H = \sum_{\mu,\nu=\pm,0,\mu\geq\nu} c^{\mu\nu} j_\mu j_\nu + \sum_{\mu=\pm,0} c^\mu j_\mu \tag{7}$$

where the coefficients $c^{\mu\nu}, c^{\mu}$ are arbitrary real numbers.

The crucial point in order to relate the operator (7) to QES problems is the introduction of the nonnegative integer $2j$ in (4). Indeed, the generators of $sl(2,R)$ as written in (4) preserve the space of polynomials of order $2j$

$$P(2j) = \{1, x, x^2, ..., x^{2j}\} \tag{8}$$

and so do the Hamiltonian (7). Searching for the eigenvalues of (1) is thus limited to the diagonalization of (7) in the $(2j+1)$-dimensional space $P(2j)$. It is a straightforward problem leading to the knowledge of the eigenvalues $E_k (k = 0, 1, ..., 2j)$ as well as the corresponding eigenfunctions ψ_k (cf. (3) where ϕ_k belongs to $P(2j)$) and ensuring the quasi-exactly solvability of the original equation.

We propose in this paper to take a new look at this problem through the consideration of the so-called polynomial deformations [3] of $sl(2,R)$ i.e. of the structures characterized by the following commutation relations

$$[J_0, J_\pm] = \pm J_\pm, \tag{9}$$

$$[J_+, J_-] = p_n(J_0), \tag{10}$$

where $p_n(J_0)$ stands for a polynomial of order n in the operator J_0.

More precisely, in Section 2, we show how to introduce in a natural manner the polynomial deformations (10) inside the operator (7) (n will then be limited to 3). In Section 3, we study the finite-dimensional representations of these polynomial deformations while Section 4 is devoted to their finite-dimensional differential realizations. In Section 5, we analyze two examples i.e. the sextic oscillator (Subsection 5.1) and the second harmonic generation (SHG) problem (Subsection 5.2). Finally, we give some conclusions in Section 6.

2 Polynomial deformations of $sl(2,R)$ inside QES problems

Instead of considering the operators (4) as expressed in (7), let us introduce the following operators (so defined for natural reasons in connection with their respective raising, diagonal or lowering characteristics)

$$J_+ \equiv c^{++} j_+^2 + c^{+0} j_+ j_0 + c^+ j_+, \tag{11}$$

$$J_0 \equiv c^{+-} j_+ j_- + c^{00} j_0^2 + c^0 j_0, \tag{12}$$

$$J_- \equiv c^{0-} j_0 j_- + c^{--} j_-^2 + c^- j_-, \tag{13}$$

so that H simply writes

$$H = J_+ + J_0 + J_-. \tag{14}$$

As we will see, restoring the linearity inside (7) such that it becomes the combination (14) will have the consequence of introducing nonlinearity inside (6) so that we will be concerned with the algebra (9)-(10). Indeed asking for the relations (9) to be satisfied with the operators (11)-(13) and the relations (5)-(6) lead to two cases only i.e. either

$$c^{++} = c^{--} = 0, c^{+-} = c^{00}, c^0 + c^{00} = 1 \tag{15}$$

or

$$c^{++} \neq 0, c^{--} \neq 0, c^{+0} = c^+ = c^{0-} = c^- = 0, c^{+-} = c^{00}, c^0 + c^{00} = \frac{1}{2}. \tag{16}$$

The relation (10) is then ensured with the respective polynomials $p_{n=3}(J_0)$

$$\begin{aligned} &p_3(J_0) = 4c^{+0}c^{0-}J_0^3 + 3(c^{0-}c^+ + c^{+0}c^- - c^{+0}c^{0-})J_0^2 \\ &+[2c^+c^- - c^{+0}c^- - c^{0-}c^+ + c^{+0}c^{0-}(1-2j(j+1))]J_0 \\ &+j(j+1)(c^{+0}c^{0-} - c^{+0}c^- - c^{0-}c^+) \end{aligned} \tag{17}$$

or

$$p_3(J_0) = 8c^{++}c^{--}((2j^2+2j-1)J_0 - 8J_0^3). \tag{18}$$

Notice that in these expressions, $c^{+-}(= c^{00})$ has been put equal to zero without loosing generality (cf. the Casimir operator of $sl(2,R)$). Moreover, the relation (18) refers to the Higgs algebra already intensively visited [3].

We can thus consider the QES Hamiltonians as linear combinations of operators generating the following polynomial deformation of $sl(2,R)$

$$[J_0, J_\pm] = \pm J_\pm, \tag{19}$$

$$[J_+, J_-] = \alpha J_0^3 + \beta J_0^2 + \gamma J_0 + \delta,\ \alpha, \beta, \gamma, \delta \in R \tag{20}$$

taking account of the two possibilities (17) and (18). Notice that the Casimir operator of this deformed algebra is

$$C = J_+J_- + \frac{\alpha}{4}J_0^4 + (\frac{\beta}{3} - \frac{\alpha}{2})J_0^2 + (\frac{\alpha}{4} - \frac{\beta}{2} + \frac{\gamma}{2})J_0^2 + (\frac{\beta}{6} - \frac{\gamma}{2} + \delta)J_0$$

.

The next step will be the determination of the finite-dimensional representations of the algebra (19)-(20) denoted in the following by $sl^{(3)}(2,R)$, the upper index referring to the highest power of the diagonal operator.

3 Finite-dimensional representations of $sl^{(3)}(2,R)$

As stated in the Introduction and in relation with the possible diagonalization of H, we are interested in the finite-dimensional ($=2J+1$, $J=0,\frac{1}{2},1,...$) representations of $sl^{(3)}(2,R)$, only. We thus consider kets of type $\mid J,M>$ with M running from $-J$ to J and such that

$$J_0 \mid J,M >= (\frac{M}{q}+c) \mid J,M >, \tag{21}$$

$$J_+ \mid J,M >= f(M) \mid J,M+q >, \tag{22}$$

$$J_- \mid J,M >= g(M) \mid J,M-q >, \tag{23}$$

where q is a positive integer and c a real number. The relations (21)-(23) are such that (19) is satisfied. In order to ensure (20), we have to impose

$$f(M-q)g(M) - f(M)g(M+q) = \alpha(\frac{M}{q}+c)^3 + \beta(\frac{M}{q}+c)^2$$
$$+\gamma(\frac{M}{q}+c) + \delta,\ M=-J,...,J. \tag{24}$$

Moreover, we have to take account of the dimension of the representations, leading to the constraints

$$f(J) = f(J-1) = ... = f(J-q+1) = 0 \tag{25}$$

and

$$g(-J) = g(-J+1) = ... = g(-J+q-1) = 0. \tag{26}$$

Being interested in the highest weight representations, we obtain from (24) and (25) the following result

$$f(J-(k+1)q-l)g(J-kq-l) = (k+1)\{\alpha(\frac{J-l}{q}+c)^3 + \beta(\frac{J-l}{q}+c)^2$$

$$+\gamma(\frac{J-l}{q}+c)+\delta-\frac{1}{2}[3\alpha(\frac{J-l}{q}+c)^2+2\beta(\frac{J-l}{q}+c)+\gamma]k \tag{27}$$
$$+\frac{1}{6}[3\alpha(\frac{J-l}{q}+c)+\beta]k(2k+1)-\frac{\alpha}{4}k^2(k+1)\}$$

where $l = 0, 1, ..., q-1$ and $k = 0, 1, ..., \frac{2J-d-l}{q}$. The nonnegative integer d introduced in the last formula has to take specific values according to l but also to J. These values are summarized in the following table, n denoting a nonnegative integer.

Table	$l=0$	$l=1$	$l=2$	...	$l=q-1$
$J=(qn)/2$	$d=0$	$d=q-1$	$d=q-2$	...	$d=1$
$J=(qn+1)/2$	$d=1$	$d=0$	$d=q-1$	...	$d=2$
$J=(qn+2)/2$	$d=2$	$d=1$	$d=0$	...	$d=3$
...	...	...	...	...	...
$J=(qn+q-1)/2$	$d=q-1$	$d=q-2$	$d=q-3$	...	$d=0$

Taking care of these values, we also have to constrain the real c in (21) and (27) through the conditions (26). This leads to equal to zero the expression inside the brackets $\{\ .\ \}$ in (27) for $k = \frac{2J-d-l}{q}$ or, in other words, to consider the q equations on c

$$\alpha[c^3+\frac{3}{2q}(d-l)c^2+\frac{1}{q^2}(J^2-J(d+l)+l^2-dl+d^2)c+\frac{1}{2q}(2J-d-l)c$$
$$+\frac{1}{4q^3}(2J^2(d-l)-2J(d^2-l^2)+d^3-d^2l+dl^2-l^3)+\frac{1}{4q^2}(l^2-d^2+2J(d-l))]$$
$$+\beta[c^2+\frac{1}{q}(d-l)c+\frac{1}{3q^2}(J^2-J(d+l)+d^2-dl+l^2)+\frac{1}{6q}(2J-d-l)]$$
$$+\gamma(c+\frac{1}{2q}(d-l))+\delta=0\ ,\ l=0,1,...,q-1. \tag{28}$$

Excluding the trivial case $(\alpha, \beta, \gamma, \delta) = (0, 0, 0, 0)$, we notice that these equations reduce to

$$\alpha c(c^2+J(J+1))+\beta(c^2+\frac{1}{3}J(J+1))+\gamma c+\delta=0, \tag{29}$$

for $q = 1$ while for $q = 2$, we are led to either

$$\alpha=0\rightarrow\beta=0, c=-\frac{\delta}{\gamma} \tag{30}$$

or

$$\alpha \neq 0 \to \delta = \frac{\beta\gamma}{3\alpha} - \frac{2\beta^3}{27\alpha^2}, c = -\frac{\beta}{3\alpha} \tag{31}$$

if $J = n$ and to

$$\alpha(3c^2 + \frac{1}{4}J(J+1) - \frac{1}{8}) + 2\beta c + \gamma = 0, \tag{32}$$

$$\alpha(c^3 + \frac{1}{4}J(J+1)c) + \beta(c^2 + \frac{1}{12}J(J+1)) + \gamma c + \delta = 0, \tag{33}$$

if $J = n + \frac{1}{2}$. Two other values of q are also available namely $q = 3$ and $q = 4$. We respectively obtain

$$\alpha \neq 0, \gamma = \frac{\beta^2}{3\alpha} - \frac{1}{9}\alpha J^2 + \frac{2}{9}\alpha, \delta = \frac{\beta\gamma}{3\alpha} - \frac{2\beta^3}{27\alpha^2}, c = -\frac{\beta}{3\alpha} \tag{34}$$

if $J = \frac{3n}{2}$,

$$\alpha \neq 0, \gamma = \frac{\beta^2}{3\alpha} - \frac{1}{9}\alpha J^2 - \frac{2}{9}\alpha J + \frac{1}{9}\alpha, \delta = \frac{\beta\gamma}{3\alpha} - \frac{2\beta^3}{27\alpha^2}, c = -\frac{\beta}{3\alpha} \tag{35}$$

if $J = \frac{3n+1}{2}$ and

$$\alpha \neq 0, \gamma = \frac{\beta^2}{3\alpha} - \frac{1}{9}\alpha J^2 - \frac{1}{9}\alpha J + \frac{1}{9}\alpha, \delta = \frac{\beta\gamma}{3\alpha} - \frac{2\beta^3}{27\alpha^2}, c = -\frac{\beta}{3\alpha} \tag{36}$$

if $J = \frac{3n+2}{2}$, these three contexts being associated with $q = 3$. We also have

$$\alpha \neq 0, \gamma = \frac{\beta^2}{3\alpha} - \frac{1}{16}\alpha J^2 + \frac{3}{16}\alpha, \delta = \frac{\beta\gamma}{3\alpha} - \frac{2\beta^3}{27\alpha^2}, c = -\frac{\beta}{3\alpha} \tag{37}$$

if $J = 2n$ and

$$\alpha \neq 0, \gamma = \frac{\beta^2}{3\alpha} - \frac{1}{16}\alpha J^2 - \frac{1}{8}\alpha J + \frac{1}{8}\alpha, \delta = \frac{\beta\gamma}{3\alpha} - \frac{2\beta^3}{27\alpha^2}, c = -\frac{\beta}{3\alpha} \tag{38}$$

if $J = 2n+1$, these two cases being related with $q = 4$. The two other systems related to $q = 4$ i.e. those corresponding to $J = 2n + \frac{1}{2}$ and $J = 2n + \frac{3}{2}$ are incompatible ones as are those related to $q > 4$.

Let us also notice that some of these representations are reducible. For instance, if we consider the case of the usual $sl(2, R)$ algebra, corresponding to $\alpha = 0, \beta = 0, \gamma = 2, \delta = 0$, we obtain

$$q = 1 \to c = 0 \tag{39}$$

and

$$q = 2 \to c = 0, J = n. \tag{40}$$

The first case (39) is associated to (see (27))

$$f(J-k-1)g(J-k) = (k+1)\{2J-k\}\ ,\ k = 0,1,...,2J \tag{41}$$

or, in other words, to

$$f(M-1)g(M) = (J-M+1)(J+M), M = -J,...,J. \tag{42}$$

We recognize in (42) the well known result of the angular momentum theory [4] subtended by this $sl(2,R)$ algebra. The second case (40) corresponds to

$$f(J-2k-2)g(J-2k) = (k+1)\{J-k\}\ ,\ k = 0,1,...,J, \tag{43}$$

$$f(J-2k-3)g(J-2k-1) = (k+1)\{J-k-1\}\ ,\ k = 0,1,...,J-1, \tag{44}$$

or, in other words, to

$$f(M-2)g(M) = \frac{1}{4}(J-M+2)(J+M), M = -J, -J+2,...,J, \tag{45}$$

$$f(M-2)g(M) = \frac{1}{4}(J-M+1)(J+M-1), M = -J+1, -J+3,...,J-1. \tag{46}$$

It is then clear that the representation $(J = n, q = 2)$ is in fact the direct sum of the two (irreducible) representations $(J = \frac{n-1}{2}, q = 1)$ and $(J = \frac{n}{2}, q = 1)$ (the eigenvalues of the Casimir being equal).

4 Finite-dimensional differential realizations of $sl^{(3)}(2,R)$

We now turn to the construction of the differential realizations (expressed in terms of the real variable x) of the algebra (19)-(20). In correspondence with (21)-(23), the generators of $sl^{(3)}(2,R)$ have the following forms

$$J_+ \equiv x^q F(D), \tag{47}$$

$$J_0 \equiv \frac{1}{q}(D-J) + c, \tag{48}$$

$$J_- \equiv G(D)\frac{d^q}{dx^q} \tag{49}$$

and the basis $\{| \, J, M >, M = -J, ..., J\}$ stands for the space $P(2J)$ (cf. (8)) of monomials $\{x^{J+M}, M = -J, ..., J\}$. Moreover, D is the dilation operator

$$D \equiv x\frac{d}{dx}. \tag{50}$$

By remembering that

$$\frac{d^q}{dx^q}x^q = \prod_{k=1}^{q}(D+k) \equiv \frac{(D+q)!}{D!} \tag{51}$$

and

$$x^q\frac{d^q}{dx^q} = \prod_{k=0}^{q-1}(D-k) \equiv \frac{D!}{(D-q)!}, \tag{52}$$

the relation (20) gives

$$\begin{aligned} &F(D-q)G(D-q)\frac{D!}{(D-q)!} - F(D)G(D)\frac{(D+q)!}{D!} = \\ &\alpha[\frac{1}{q}(D-J)+c]^3 + \beta[\frac{1}{q}(D-J)+c]^2 \\ &+\gamma[\frac{1}{q}(D-J)+c] + \delta. \end{aligned} \tag{53}$$

Let us discuss this condition within the ($\alpha \neq 0$)-case first and the ($\alpha = 0$)-case second. a) In order to avoid singularities, we thus impose, when $\alpha \neq 0$,

$$F(D)G(D) = -\frac{\alpha}{4q^4}\frac{D!}{(D+q)!}(D+\lambda_1)(D+\lambda_2)(D+\lambda_3)(D+\lambda_4) \tag{54}$$

and obtain the following system on the real unknows $\lambda_1, ..., \lambda_4$

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 4qc + 2q - 4J + \frac{4}{3}\frac{\beta}{\alpha}q, \tag{55}$$

$$\begin{aligned} &\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 = 6q^2c^2 + 6q^2c - 12qJc \\ &+4\frac{\beta}{\alpha}q^2c + 6J^2 - 6qJ - 4\frac{\beta}{\alpha}qJ + q^2 + 2\frac{\beta}{\alpha}q^2 + 2\frac{\gamma}{\alpha}q^2, \end{aligned} \tag{56}$$

$$\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_1\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_4=$$
$$4q^3c^3-12q^2Jc^2+6q^3c^2+4\frac{\beta}{\alpha}q^3c^2+12qJ^2c-12q^2Jc$$
$$+2q^3c-8\frac{\beta}{\alpha}q^2Jc+4\frac{\beta}{\alpha}q^3c+4\frac{\gamma}{\alpha}q^3c-4J^3+6qJ^2-2q^2J$$
$$+4\frac{\beta}{\alpha}qJ^2-4\frac{\beta}{\alpha}q^2J+\frac{2}{3}\frac{\beta}{\alpha}q^3-4\frac{\gamma}{\alpha}q^2J+2\frac{\gamma}{\alpha}q^3+4\frac{\delta}{\alpha}q^3. \tag{57}$$

Let us recall that we are interested in finite-dimensional (=$2J+1$) realizations only. This means that the cases $q=1$ and $q=2$ are the only possibilities in accordance with

$$q=1\rightarrow\lambda_1=1,\lambda_2=-2J \tag{58}$$

and

$$q=2\rightarrow\lambda_1=1,\lambda_2=2,\lambda_3=-2J,\lambda_4=-2J+1. \tag{59}$$

In the first context ($q=1$), the equations (55) and (56) fix λ_3 and λ_4 as follows

$$\lambda_3=2c+\frac{1}{2}-J+\frac{2}{3}\frac{\beta}{\alpha}+\frac{\epsilon}{2},\lambda_4=2c+\frac{1}{2}-J+\frac{2}{3}\frac{\beta}{\alpha}-\frac{\epsilon}{2} \tag{60}$$

with

$$\epsilon^2=1+\frac{16}{9}\frac{\beta^2}{\alpha^2}-4J(J+1)-8c^2-\frac{16}{3}\frac{\beta}{\alpha}c-8\frac{\gamma}{\alpha} \tag{61}$$

while the equation (57) coincides with (29). In the second context ($q=2$), we are led to (31) supplemented by

$$\gamma=\frac{\beta^2}{3\alpha}-\frac{\alpha}{4}J^2-\frac{\alpha}{4}J+\frac{\alpha}{8}. \tag{62}$$

b) The case $\alpha=0$ is more simple and has already been analyzed [5]. For self-consistency, we recall the main results i.e.

$$F(D)G(D)=-\frac{\beta}{3q^3}\frac{D!}{(D+q)!}(D+\lambda_1)(D+\lambda_2)(D+\lambda_3) \tag{63}$$

where the only possible finite-dimensional (=$2J+1$) realization is associated with $q=1$ and corresponds to

$$\lambda_1=1,\lambda_2=-2J,\lambda_3=-J+3c+\frac{3\gamma}{2\beta}+\frac{1}{2}, \tag{64}$$

the real c being fixed through

$$\beta c^2 + \gamma c + \delta + \frac{\beta}{3} J(J+1) = 0. \tag{65}$$

Now that both cases have been considered, let us conclude this Section by noticing that some realizations (namely the ones corresponding to $q = 2$ without the condition (62) and the ones associated with $q = 3, 4$) are missing with respect to the representations developed in the previous Section. Indeed, the relation (53) we have imposed is more constraining because it is a relation between operators independently of the basis on which they are supposed to act. If we take account of this basis i.e. in this case $P(2J) = \{x^{J+M}, M = -J, ..., J\}$, we can recover all the cases previously discussed. For example, in the context $q = 3$, $J = \frac{3}{2}$, we can consider

$$J_+ = -\frac{1}{6} f(-\frac{3}{2}) x^3 (D-1)(D-2)(D-3), \tag{66}$$

$$J_0 = \frac{1}{3} D - \frac{1}{2} - \frac{\beta}{3\alpha}, \tag{67}$$

$$J_- = \frac{1}{6} g(\frac{3}{2}) \frac{d^3}{dx^3} \tag{68}$$

with

$$f(-\frac{3}{2}) g(\frac{3}{2}) = \frac{\alpha}{9}. \tag{69}$$

It is then easy to convince ourselves that these operators generate $sl^{(3)}(2, R)$ with $\gamma = \frac{\beta^2}{3\alpha} - \frac{\alpha}{36}$ and $\delta = \frac{\beta^3}{27\alpha^2} - \frac{\beta}{108}$ but on the space $P(3)$ only (the relation (53) being trivially not satisfied except on this space). However it has to be stressed that the relations corresponding to (66)-(68) but in the general context become really heavy when the value of J increases.

5 Two examples

We first consider the prototype of QES systems i.e. the so-called sextic oscillator [1,2] and then turn to a more physical example: the SHG problem.

5.1 The sextic oscillator

This system is characterized by the following potential

$$V(y) = a^2y^6 + 2aby^4 + (b^2 - 2ap - 8aj - 3a)y^2 \tag{70}$$

with $a(\neq 0), b \in R$ and $p = 0, 1$ while j is the quantum number appearing in (4). With

$$x = y^2 \tag{71}$$

and

$$\chi = -\int (\frac{a}{2}x + \frac{b}{2} - \frac{p}{2x})dx, \tag{72}$$

we can be convinced of its QES character via the form (7) (up to a translation)

$$H = J_+ + J_0 + J_- + 2bp + 4bj + b \tag{73}$$

and

$$c^{0-} = -4, c^+ = -4a, c^0 = 4b, c^- = -(4j + 2 + 4p), \tag{74}$$

the other $c's$ being equal to zero. Without loss of generality, we can put $b = \frac{1}{4}$ (in order to recover (19)) and obtain, through (11)-(13), the relation (20) with

$$\alpha = 0, \beta = 48a, \gamma = 32a(p + j), \delta = -16aj(j + 1). \tag{75}$$

The case $q = 1$ is thus the only one to be available. We actually have

$$J_0 \mid J, M >= (M + c) \mid J, M >, \tag{76}$$

$$J_+ \mid J, M >= f(M) \mid J, M + 1 >, \tag{77}$$

$$J_- \mid J, M >= g(M) \mid J, M - 1 >, \tag{78}$$

with

$$f(M-1)g(M) = (J-M+1)(J+M)(48ac+16aM+16ap+16aj-8a). \tag{79}$$

Moreover, the parameter c is fixed according to

$$c = -\frac{1}{3}(p + j) \pm \frac{1}{3}\sqrt{(p + j)^2 - 3J(J + 1) + 3j(j + 1)} \tag{80}$$

leading to constrain J through

$$J \leq -\frac{1}{2} + \frac{1}{6}\sqrt{36j(j+1) + 12(p+j)^2 + 9} \tag{81}$$

in order to ensure the reality of c. Because the space $P(2J)$ is preserved, we just have to equal to zero the determinant of the following matrix

$$\begin{aligned} \mathrm{M} = & \sum_{\mathrm{k}=1}^{2\mathrm{J}+1} (\mathrm{E} + \mathrm{J} - \mathrm{k} - \mathrm{c} - \frac{\mathrm{p}}{2} - \mathrm{j} + \frac{3}{4}) \mathrm{e}_{\mathrm{k,k}} \\ & - \sum_{k=1}^{2J} g(-J+k) e_{k,k+1} - \sum_{k=1}^{2J} f(-J+k-1) e_{k+1,k} \end{aligned} \tag{82}$$

in order to find the energies. In the matrix (82), the notation $e_{k,l}$ stands for a $(2J+1)$-dimensional matrix where 1 is at the intersection of the k^{th} row and the l^{th} column, the other elements being 0. For example, when $j = \frac{1}{2}$, we have

$$J = 0, \frac{1}{2}, \tag{83}$$

according to (81) while the relation (80) gives

$$c = -\frac{p}{3} - \frac{1}{6} \pm \frac{1}{3}\sqrt{(p+\frac{1}{2})^2 + \frac{9}{4}} \tag{84}$$

if $J = 0$ and

$$c = 0 \ \ or \ \ c = -\frac{1}{3}(2p+1) \tag{85}$$

if $J = \frac{1}{2}$. In the case (84), the energies are

$$E = c + \frac{p}{2} + \frac{3}{4} \rightarrow E = 0.0428932; 0.0562872; 1.1103796; 1.4571067 \tag{86}$$

and in the case (85), the resolution of the vanishing determinant associated with (82) leads to

$$E = \frac{3}{4} \pm \frac{1}{2}\sqrt{1+32a}; E = \frac{5}{4} \pm \frac{1}{2}\sqrt{1+96a} \tag{87}$$

if $c = 0$ and

$$E = \frac{5}{12} \pm \frac{1}{2}\sqrt{1-32a}; E = \frac{1}{4} \pm \frac{1}{2}\sqrt{1-96a} \tag{88}$$

if $c=-\frac{1}{3}$; $c=-1$. Only the values given in (87) correspond to the previously obtained ones [2]. This is indeed a general result that the known energies [2] are recovered through our approach when $c=0, J=j$. The other contexts $(J<j)$ or $(J=j, c=-\frac{2}{3}(p+j))$ lead to supplementary new values of the energy.

Let us analyze more deeply this result by going to the differential realization (47)-(49) i.e.

$$J_+ = xF(D), \tag{89}$$

$$J_0 = D - J + c, \tag{90}$$

$$J_- = G(D)\frac{d}{dx}, \tag{91}$$

where (cf. (63) and (64))

$$F(D)G(D) = -16a(D-2J)(D-J+3c+\frac{1}{2}+p+j). \tag{92}$$

In order to preserve the space $P(2J)$, let us make the choice (without loss of generality, this freedom being due to the fact that $sl^{(3)}(2,R)$ is defined up to an automorphism [5])

$$G(D) = -4(D-J+3c+\frac{1}{2}+p+j). \tag{93}$$

In that case, the Hamiltonian (73) is realized as

$$\begin{aligned} H = & -4x\frac{d^2}{dx^2} + [4ax^2 + x - 4(-J+3c+\frac{1}{2}+p+j)]\frac{d}{dx} \\ & -8aJx + \frac{p}{2} + \frac{1}{4} + j - J + c. \end{aligned} \tag{94}$$

This form is analog to (2) and we propose to write it in the Schrödinger form (1) through the changes (3) i.e.

$$x = y^2, \tag{95}$$

$$\psi = exp(\int(\frac{1}{8} - 2ax + \frac{1}{2}(J-3c-p-j)\frac{1}{x})dx)\phi. \tag{96}$$

The potential obtained in this manner is given by

$$V(y) = a^2y^6 + \frac{1}{2}ay^4 + (\frac{1}{16} - 6aJ - 2aj - 6ac - 2ap - 3a)y^2$$
$$+\frac{1}{2}(j - J - c) + (J - 3c - p - j)(J - 3c - p - j + 1)\frac{1}{y^2}. \qquad (97)$$

Compared with (70), this expression actually reduces to the sextic oscillator potential iff $c = 0$ and $J = j$. For other values of the parameters (i.e. the already cited $(J < j)$ and $(J = j, c = -\frac{2}{3}(p + j))$, the new eigenvalues of the problem (appearing for $j = \frac{1}{2}$ in (86) and (88)) do correspond to another model, namely the *radial* sextic oscillator as shown by (97).

5.2 The second harmonic generation

This nonlinear optical process as well as others such as coherent spontaneous emission and down conversion [6] can be described by the following effective Hamiltonian

$$H = a_1^\dagger a_1 + 2a_2^\dagger a_2 + g(a_2^\dagger a_1^2 + (a_1^\dagger)^2 a_2) \qquad (98)$$

with cubic terms in the (independent) bosonic creation and annihilation operators. The Hamiltonian (98) has already been recognized [7] as a QES model. We confirm such a result by using the technique developed in Section 2. Indeed following Section 2, we propose to define

$$J_+ = a_2^\dagger a_1^2, \qquad (99)$$

$$J_0 = \frac{1}{3}(a_2^\dagger a_2 - a_1^\dagger a_1), \qquad (100)$$

$$J_- = (a_1^\dagger)^2 a_2 \qquad (101)$$

such that the algebra (19)-(20) is satisfied with

$$\alpha = 0, \beta = -12, \gamma = 0, \delta = \frac{1}{3}N^2 + N. \qquad (102)$$

In the last expression, N is the invariant

$$N = a_1^\dagger a_1 + 2a_2^\dagger a_2 \qquad (103)$$

satisfying

$$[N, J_0] = [N, J_\pm] = 0. \tag{104}$$

Once again, the values (102) are typical of the $q = 1$-representation only so that the relations (76)-(78) are the ones to be taken care of but with

$$f(M-1)g(M) = (J-M+1)(J+M)(-12c-4M+2). \tag{105}$$

The parameter c is fixed according to

$$c^2 = -\frac{1}{3}J(J+1) + \frac{1}{36}N^2 + \frac{1}{12}N \tag{106}$$

and its reality is ensured if

$$J \leq -\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{3}N^2 + N + 1}. \tag{107}$$

It is then possible to determine the energies by putting to zero the determinant of a $(2J+1)$ by $(2J+1)$ matrix analog to (82) as well as it is possible to determine them through the differential realization (89)-(91). In this case, we have

$$F(D)G(D) = 4(D-2J)(D-J+3c+\frac{1}{2}) \tag{108}$$

and choosing

$$G(D) = 1 \tag{109}$$

the Hamiltonian (98) becomes

$$\begin{aligned} H &= N + 4gx^3\frac{d^2}{dx^2} + g(1+12(-J+c+\frac{1}{2})x^2)\frac{d}{dx} \\ &+4g(-J+2J^2-6Jc)x. \end{aligned} \tag{110}$$

With the respective changes of variables and wavefunctions

$$x = -\frac{1}{gy^2}, \psi = exp(-\int (\frac{1}{8x^3} + \frac{3}{2}(c-J)\frac{1}{x})dx)\phi \tag{111}$$

we can put (110) on the Schrödingerlike form (1) with

$$V(y) = \frac{g^4}{16}y^6 + \frac{3}{2}g^2(c-J-\frac{1}{2})y^2 + (J+3c)(J+3c+1)\frac{1}{y^2} + N. \tag{112}$$

Once again this is typical of a radial sextic oscillator and the QES characteristics of the second harmonic generation is thus proved. The determination of the energies is then possible without any difficulty [2]. For example, when $N = 4$, we have

$$J = 0 \rightarrow E = 4, \tag{113}$$

$$J = \frac{1}{2} \rightarrow E = 4 \pm g\sqrt{2\sqrt{19}}, \tag{114}$$

$$J = 1 \rightarrow E = 4, 4 \pm 4g. \tag{115}$$

Only the values (115) correspond to known ones, the values (113), (114) coming from other models. This is a general result in the sense that the energies of SHG are the ones of the Schrödinger potential (112) with

$$c = J - \frac{N}{3} \tag{116}$$

and

$$J = \frac{N}{4} \ \ or \ \ J = \frac{N-1}{4} \tag{117}$$

according to even or odd values of N. The SHG potential thus writes

$$V(y) = \frac{g^4}{16}y^6 - \frac{g^2}{4}(2N+3)y^2 + N \tag{118}$$

and exactly coincides with the potential (42) of Ref. [7]. In terms of the operators (4) (with $j = J$), this gives

$$H = gj_+(j_0 + \frac{1}{2} - \frac{N}{4}) - 4gj_- + N \tag{119}$$

if N is even and

$$H = gj_+(j_0 - \frac{1}{4} - \frac{N}{4}) - 4gj_- + N \tag{120}$$

if N is odd.

6 Conclusions

We have developed a general method based on the polynomial deformations of the Lie algebra $sl(2, R)$ in order to exhibit the QES characteristics of a Hamiltonian. We have applied this method to two examples: one more theoretical -the sextic oscillator- and one more physical -the second harmonic generation-. In both cases, a finite number of energies as well as eigenfunctions are determined through the finite-dimensional representations -or, in an equivalent way, through the realizations- of these polynomial deformations. Some of these energies (and eigenfunctions) were previously known, not the others. It seems, through the analysis of the two examples, that these previously unknown energies do not correspond to the same model but to another one being closely related to the first one. In some cases, the comparison between these new and old models could be interesting, with respect to experimental data in particular.

The main advantage of the method we have proposed is that it can be systematically applied to any Hamiltonian written in terms of a raising and a lowering operator. Numerous physical Hamiltonians are of that type. In particular, we plan to analyze one of them: the so-called Lipkin-Meshkov-Glick Hamiltonian [8] of specific interest in nuclear physics. Being based on a ($\alpha \neq 0$) polynomial deformation, its analysis is more delicate but also richer in the number of available representations [9].

The main drawback is that it is limited to $sl(2, R)$ and its deformations when we know that some QES models need more extended algebras. However, the method we have presented here can also be generalized to these extended algebras as well as superalgebras. We also plan to come back on these points in the near future.

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